

Optimal bounds for convergence of expected spectral distributions to the semi-circular law for the $4 + \varepsilon$ moment ensemble

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Abstract

This paper extends a previous bound of order $O(n^{-1})$ of the authors [1] for the rate of convergence in Kolmogorov distance of the expected spectral distribution of a Wigner random matrix ensemble to the semicircular law. We relax the moment conditions for entries of the Wigner matrices from order 8 to order $4 + \varepsilon$ for an arbitrary small $\varepsilon > 0$.

1 Introduction

Consider a family $\mathbf{X} = \{X_{jk}\}$, $1 \leq j \leq k \leq n$, of independent real random variables defined on some probability space $(\Omega, \mathfrak{M}, \Pr)$, for any $n \geq 1$. Assume that $X_{jk} = X_{kj}$, for $1 \leq k < j \leq n$, and introduce the symmetric matrices

$$\mathbf{W} = \frac{1}{\sqrt{n}} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}.$$

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The matrix \mathbf{W} has a random spectrum $\{\lambda_1, \dots, \lambda_n\}$ and an associated spectral distribution function $\mathcal{F}_n(x) = \frac{1}{n} \text{card}\{j \leq n : \lambda_j \leq x\}$, $x \in \mathbb{R}$. Averaging over the random values $X_{ij}(\omega)$, define the expected (non-random) empirical distribution functions $F_n(x) = \mathbf{E} \mathcal{F}_n(x)$. Let $G(x)$ denote the semi-circular distribution function with density $g(x) = G'(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{I}_{[-2, 2]}(x)$, where $\mathbb{I}_{[a, b]}(x)$ denotes the indicator-function of the interval $[a, b]$. Let $\Delta_n := \sup_x |F_n(x) - G(x)|$. In a recent paper [1] we proved the following result

Theorem 1.1. *Let $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$. Assume that for some $0 < \varkappa \leq 4$*

$$\sup_{n \geq 1} \sup_{1 \leq j, k \leq n} \mathbf{E}|X_{jk}|^4 =: \mu_4 < \infty. \quad (1.1)$$

Assume as well that there exists a constant D_0 such that for all $n \geq 1$

$$\sup_{1 \leq j, k \leq n} |X_{jk}| \leq D_0 n^{\frac{1}{4}}. \quad (1.2)$$

Then, there exists a positive constant $C = C(D_0, \mu_4)$ depending on D_0 and μ_4 only such that

$$\Delta_n = \sup_x |F_n(x) - G(x)| \leq C n^{-1}. \quad (1.3)$$

Corollary 1.1. *Let $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$. Assume that*

$$\sup_{n \geq 1} \sup_{1 \leq j, k \leq n} \mathbf{E}|X_{jk}|^8 =: \mu_8 < \infty. \quad (1.4)$$

Then, there exists a positive constant $C = C(\mu_8)$ depending on μ_8 only such that

$$\Delta_n \leq C n^{-1}. \quad (1.5)$$

Here we describe some refinements of the proof of Theorem 1.3 in [1] showing that

Theorem 1.2. *Assume that for some $0 < \varkappa \leq 4$ there exists positive constant $0 < \mu_{4+\varkappa} < \infty$ such that*

$$\sup_{j, k \leq 1} \mathbf{E}|X_{jk}|^{4+\varkappa} \leq \mu_{4+\varkappa}. \quad (1.6)$$

Then there exists a positive constant C depending on \varkappa and $\mu_{4+\varkappa}$ only such that

$$\Delta_n \leq C n^{-1}. \quad (1.7)$$

Using standard techniques similar to [1], we may reduce the problem to the following

Theorem 1.3. *Assume that there exists a constant $0 < \mu_{4+\varkappa} < \infty$ such that*

$$\sup_{j,k \geq 1} \mathbf{E}|X_{jk}|^{4+\varkappa} \leq \mu_{4+\varkappa}. \quad (1.8)$$

Assume that for $\alpha := \frac{2}{4+\varkappa}$

$$|X_{jk}| \leq Dn^\alpha. \quad (1.9)$$

Then there exists a positive constant $C = C(D, \mu_{4+\varkappa}, \varkappa)$ depending on D , $\mu_{4+\varkappa}$ and \varkappa only such that

$$\Delta_n = \sup_x |F_n(x) - G(x)| \leq Cn^{-1}. \quad (1.10)$$

For a discussion of this and previous results the reader should consult the introduction of [1].

For the proof we need to revise parts of the proof of Theorem 1.3 in [1]. The conditions of Theorem 1.2 suffice for the remaining parts and only the result [1][Theorem 1.3] needs to be strengthened. Let us reformulate this theorem here. Given constants $a > 0$, $A_0 > 0$, (to be chosen later), let $\frac{1}{2} > \varepsilon > 0$ be a sequence of positive numbers (depending on n) such that

$$\varepsilon^{\frac{3}{2}} = 2v_0a, \quad v_0 := A_0n^{-1}. \quad (1.11)$$

Define the region

$$\mathbb{G} = \{z = u + iv : |u| \leq 2 - \varepsilon, v\sqrt{\gamma} \geq v_0\}, \quad \gamma := \gamma(z) = |2 - |u||, \quad \varepsilon = c_0v_0^{\frac{2}{3}} \quad (1.12)$$

(see as well [1, Definition (1.11)]).

Theorem 1.4. *Assuming the conditions of Theorem 1.3, there exist positive constants $A_0 > 0$ and $C = C(D, A_0, \mu_{4+\varkappa})$ depending on D , A_0 and $\mu_{4+\varkappa}$ only, such that, for $z \in \mathbb{G}$*

$$|\mathbf{E}m_n(z) - s(z)| \leq \frac{C}{nv^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|z^2 - 4|^{\frac{1}{4}}}.$$

The key ingredient for the proof of Theorem 1.3 is an essential improvement of the dependence of bounds of the quantity $\mathbf{E}|\varepsilon_{j2}|^p$ in (2.27) on v . This is due to a considerably improved estimate for off-diagonal entries of the resolvent matrix, see Lemmas 2.4 and 2.3, compared to the previous bounds in [1, Lemma 5.8].

2 Estimation of moments of diagonal resolvent entries

In what follows we shall assume $z \in \mathbb{G}$. At first let us modify the proofs of Lemma 5.13 and Corollary 5.14 of [1]. We prove the following

Theorem 2.1. *Assuming the conditions of Theorem 1.3, there exist constants C_0 , A_0 and A_1 depending on \varkappa and D such that*

$$\mathbf{E}|R_{jj}|^p \leq C_0^p, \quad (2.1)$$

for $p \leq A_1(nv)^{\frac{1-2\alpha}{2}}$ and $v \geq A_0 n^{-1}$.

In order to prove this result we shall need the following Lemmas. Let $s = 2^{\frac{2}{1-2\alpha}}$.

Lemma 2.1. *Assuming the conditions of Theorem 1.3, we have, for all $v \geq v_0$*

$$\mathbf{E}|\varepsilon_{j1}|^{2p} \leq C^p n^{-p(1-2\alpha)}. \quad (2.2)$$

Proof. Note that

$$|\varepsilon_{j1}| \leq |X_{jj}|/\sqrt{n} \leq Dn^{-\frac{1-2\alpha}{2}}. \quad (2.3)$$

This inequality concludes the proof of Lemma 2.1. \square

Lemma 2.2. *Assuming condition (2.8) for $v \geq v_1$, we have, for all $v \geq v_1/s$*

$$\mathbf{E}|\varepsilon_{j3}|^{2p} \leq C^p p^{2p} n^{-2p(1-2\alpha)} (sH_0)^{2p}. \quad (2.4)$$

Proof. Recall that

$$\varepsilon_{j3} = \frac{1}{n} \sum_{l \in \mathbb{T}_j} (X_{jl}^2 - 1) R_{ll}^{(j)}. \quad (2.5)$$

Applying Rosenthal's inequality, we get

$$\mathbf{E}|\varepsilon_{j3}|^{2p} \leq C^p \left(p^p \mu_4^p n^{-p} \mathbf{E} \left(\frac{1}{n} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^2 \right)^p + p^{2p} n^{-2p} \mu_{4p} \sum_{l \in \mathbb{T}_j} \mathbf{E}|R_{ll}^{(j)}|^{2p} \right). \quad (2.6)$$

By the assumptions of Theorem 1.3, we get

$$\mathbf{E}|\varepsilon_{j3}|^{2p} \leq C^p (sH_0)^{2p} (p^p n^{-p} + p^{2p} n^{-2p(1-2\alpha)}). \quad (2.7)$$

Here we used that $4\alpha > 1$ and $\alpha < \frac{1}{2}$. These relations conclude the proof of Lemma 2.2. \square

Lemma 2.3. *Assume that for some $1 \geq v_1 \geq v_0$ there exists a sufficiently large constant H_0 such that for any \mathbb{J} with cardinality $|\mathbb{J}| \leq \log_s(nv)$ and $q \leq A_1(nv)^{\frac{1-2\alpha}{2}}$ the inequality*

$$\mathbf{E}|R_{jk}^{(\mathbb{J})}|^q \leq H_0^q \quad (2.8)$$

holds for any $j, k \in \mathbb{T} \setminus \mathbb{J}$ and any $v \geq v_1$. Then this inequality still holds for any $v \geq v_1/s$ and $j \neq k \in \mathbb{T} \setminus \mathbb{J}$.

Proof. We use the representation

$$R_{jk}^{(\mathbb{J})} = -\left(\frac{1}{\sqrt{n}} \sum_{l \in \mathbb{T}_{\mathbb{J}}} X_{jl} R_{lk}^{(\mathbb{J},j)}\right) R_{kk}. \quad (2.9)$$

Applying Cauchy's and Rosenthal's inequalities, we get

$$\begin{aligned} \mathbf{E}|R_{jk}^{(\mathbb{J})}|^q &\leq H_0^q s^q \left(C^q n^{-\frac{q}{2}} \mathbf{E}^{\frac{1}{2}} \left(\sum_{l \in \mathbb{T}_{\mathbb{J}}} |R_{lk}^{(\mathbb{J},j)}|^2 \right)^q + C^q q^q n^{-\frac{q}{2}} \mu_{2q}^{\frac{1}{2}} \mathbf{E}^{\frac{1}{2}} |R_{kk}^{(\mathbb{J},j)}|^{2q} \right. \\ &\quad \left. + C^q q^q n^{-\frac{q}{2}} \mu_{2q}^{\frac{1}{2}} \mathbf{E}^{\frac{1}{2}} \left(\sum_{l \in \mathbb{T}_{\mathbb{J},j,k}} |R_{lk}^{(\mathbb{J},j)}|^{2q} \right) \right). \end{aligned} \quad (2.10)$$

Applying now [1, Lemma 7.6], inequality (2.21) and the assumptions of Lemma 2.2, we get

$$\begin{aligned} \mathbf{E}|R_{jk}^{(\mathbb{J})}|^q &\leq H_0^q s^q \left(C^q q^{\frac{q}{2}} n^{-\frac{q}{2}} v^{-\frac{q}{2}} \mathbf{E}^{\frac{1}{2}} (\operatorname{Im} R_{kk}^{(\mathbb{J},j)})^q + C^q q^q n^{-\frac{q}{2}(1-2\alpha)} n^{-1} H_0^q s^q \right. \\ &\quad \left. + C^q q^q n^{-\frac{q}{2}(1-2\alpha)} n^{-1} \mathbf{E}^{\frac{1}{2}} \left(\sum_{l \in \mathbb{T}_{\mathbb{J},j,k}} |R_{lk}^{(\mathbb{J},k)}|^{2q} \right) \right). \end{aligned} \quad (2.11)$$

Note that

$$|R_{lk}^{(\mathbb{J},j)}(u + isv) - R_{lk}^{(\mathbb{J},j)}(u + iv)| \leq (s-1)v |[\mathbf{R}^{(\mathbb{J},j)}(u + isv) \mathbf{R}^{(\mathbb{J},j)}(u + iv)]_{lk}|. \quad (2.12)$$

Applying the Cauchy–Schwartz inequality and [1, Lemma], we get

$$|R_{lk}^{(\mathbb{J},j)}(u + isv) - R_{lk}^{(\mathbb{J},j)}(u + iv)| \leq \sqrt{s} \sqrt{|R_{ll}^{(\mathbb{J},j)}(u + isv)| |R_{kk}^{(\mathbb{J},j)}(u + iv)|}. \quad (2.13)$$

Using now condition (2.8), we obtain

$$\mathbf{E}|R_{lk}^{(\mathbb{J},j)}(u + iv)|^q \leq 2^q \mathbf{E}|R_{lk}^{(\mathbb{J},j)}(u + isv)|^q + 2^q (sH_0)^q \leq 2^{q+1} s^q (sH_0)^q. \quad (2.14)$$

Inequalities (2.11) and (2.14) together imply

$$\mathbf{E}|R_{jk}^{(\mathbb{J})}|^q \leq H_0^q \left(\left(\frac{Cqs^3H_0}{nv} \right)^{\frac{q}{2}} + \frac{1}{\sqrt{n}} \left(\frac{4Cqs^2H_0}{n^{\frac{1-2\alpha}{2}}} \right)^q \right). \quad (2.15)$$

We may choose the constant A_0 such that for sufficiently large n

$$\mathbf{E}|R_{jk}^{(\mathbb{J})}|^q \leq H_0^q. \quad (2.16)$$

This completes the proof of Lemma 2.3. \square

Lemma 2.4. *Assuming condition (2.8) for $v \geq v_1$, we have, for all $v \geq v_0$ and $v \geq v_1/s$,*

$$\mathbf{E}|\varepsilon_{j2}|^{2p} \leq \left(\frac{Cp^4s^2H_0^2}{(nv)^{2(1-2\alpha)}} \right)^p. \quad (2.17)$$

Proof. Recall that

$$\varepsilon_{j2} = \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} X_{jl} X_{jk} R_{kl}^{(j)}. \quad (2.18)$$

Applying Lemma 4.1, we obtain

$$\begin{aligned} \mathbf{E}|\varepsilon_{j2}|^{2p} &\leq C^p n^{-2p} \left(p^{2p} \mathbf{E} \left(\sum_{l \neq k \in \mathbb{T}_j} |R_{lk}^{(j)}|^2 \right)^p + p^{3p} \mu_{2p} \sum_{k \in \mathbb{T}_j} \mathbf{E} \left(\sum_{l \in \mathbb{T}_{jk}} |R_{kl}^{(j)}|^2 \right)^p \right. \\ &\quad \left. + p^{4p} \mu_{2p}^2 \sum_{l \neq k \in \mathbb{T}_j} \mathbf{E}|R_{kl}^{(j)}|^{2p} \right). \end{aligned} \quad (2.19)$$

At first we apply [1][Lemma 7.6 inequalities (7.11), (7.12)] and condition (2.8) of Lemma 2.3 obtaining

$$\begin{aligned} \mathbf{E}|\varepsilon_{j2}|^{2p} &\leq C^p n^{-2p} (p^p n^p v^{-p} s^p H_0^p \\ &\quad + p^{3p} \mu_{2p} n v^{-p} (s_0 H_0)^p + p^{4p} \mu_{2p}^2 \sum_{l \neq k \in \mathbb{T}_j} \mathbf{E}|R_{kl}^{(j)}|^{2p}). \end{aligned} \quad (2.20)$$

Using the assumptions of Theorem 1.3 we have

$$\mu_{2p} \leq D^{2p} n^{2p\alpha} n^{-2} \mu_{4+\kappa}. \quad (2.21)$$

Combining the last two inequalities and using that $2p(1-\alpha) \geq p$, we get

$$\mathbf{E}|\varepsilon_{j2}|^{2p} \leq \left(\frac{Cp^2sH_0}{nv} \right)^p + \left(\frac{Cp^3H_0s}{nv} \right)^p + \left(\frac{Cp^4s^2H_0^2}{n^{2(1-2\alpha)}} \right)^p. \quad (2.22)$$

Using that $1/v \geq \frac{1}{4}$ and $2p(1 - 2\alpha) \leq p$ for $z \in \mathbb{G}$, we may write

$$\mathbf{E}|\varepsilon_{j2}|^{2p} \leq \left(\frac{Cp^4 s^2 H_0^2}{(nv)^{2(1-2\alpha)}} \right)^p. \quad (2.23)$$

Thus Lemma 2.4 is proved. \square

Lemma 2.5. *Assume that for some $1 \geq v_1 \geq v_0$ there exists a sufficiently large constant H_0 such that for any \mathbb{J} with cardinality $|\mathbb{J}| \leq \log_s(nv)$ and $q \leq A_1(nv)^{\frac{1-2\alpha}{2}}$ the inequality*

$$\mathbf{E}|R_{jk}^{(\mathbb{J})}|^q \leq H_0^q \quad (2.24)$$

holds for any $j, k \in \mathbb{T} \setminus \mathbb{J}$ and any $v \geq v_1$. Then

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^q \leq H_0^q \quad (2.25)$$

holds for any $v \geq v_1/s$ and $j \in \mathbb{T} \setminus \mathbb{J}$.

Proof. First we use representation (3.7) in [1]. We have

$$\mathbf{R}_{jj}^{(\mathbb{J})} = s(z) + s(z)(\varepsilon^{(\mathbb{J})} + \Lambda_n^{(\mathbb{J})})R_{jj}^{(\mathbb{J})}, \quad (2.26)$$

where $\varepsilon_j^{(\mathbb{J})} = \varepsilon_{j1}^{(\mathbb{J})} + \dots + \varepsilon_{j4}^{(\mathbb{J})}$ and

$$\begin{aligned} \varepsilon_{j1}^{(\mathbb{J})} &= \frac{1}{\sqrt{n}}X_{jj}, \quad \varepsilon_{j2}^{(\mathbb{J})} = -\frac{1}{n} \sum_{l \neq k \in \mathbb{T}_{\mathbb{J}}} X_{jl}X_{jk}R_{lk}^{\mathbb{J},j),} \\ \varepsilon_{j3}^{(\mathbb{J})} &= -\frac{1}{n} \sum_{l \in \mathbb{T}_{\mathbb{J}}} (X_{jl}^2 - 1)R_{ll}^{(\mathbb{J},j)}, \quad \varepsilon_{j4}^{(\mathbb{J})} = \frac{1}{n}(\text{Tr } \mathbf{R}^{(\mathbb{J})} - \text{Tr } \mathbf{R}^{\mathbb{J},j}). \end{aligned} \quad (2.27)$$

Equality (2.26) yields

$$|R_{jj}^{(\mathbb{J})}| \leq 1 + C(|\varepsilon_j^{(\mathbb{J})}| + |\Lambda_n^{(\mathbb{J})}(z)|)|R_{jj}^{(\mathbb{J})}|. \quad (2.28)$$

Since $|\Lambda_n| \leq C\sqrt{|T_n|}$ for $z \in \mathbb{G}$ (see [1, Lemma 5.9]), we get

$$|R_{jj}^{(\mathbb{J})}| \leq 1 + C|R_{jj}^{(\mathbb{J})}|(|\varepsilon_j^{(\mathbb{J})}| + \sqrt{|T_n|}). \quad (2.29)$$

Applying Cauchy's inequality, we get

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \leq 3^p(1 + (\mathbf{E}^{\frac{1}{2}}|\varepsilon_j^{(\mathbb{J})}|^{2p} + \mathbf{E}^{\frac{1}{2}}|T_n^{(\mathbb{J})}|^p)\mathbf{E}^{\frac{1}{2}}|R_{jj}^{(\mathbb{J})}|^{2p}). \quad (2.30)$$

Note that if $p \leq A_1(nv/s)^{\frac{1-2\alpha}{2}}$, then $2p \leq A_1(nv)^{\frac{1-2\alpha}{2}}$. Applying our assumption we get

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \leq 3^p + (sH_0)^p (\mathbf{E}^{\frac{1}{2}}|\varepsilon_j^{(\mathbb{J})}|^{2p} + \mathbf{E}^{\frac{1}{2}}|T_n^{(\mathbb{J})}|^p) \quad (2.31)$$

Consider inequality

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \leq 3^p + 3^p (sH_0)^p (\mathbf{E}^{\frac{1}{2}}|\varepsilon_j^{(\mathbb{J})}|^{2p} + \mathbf{E}^{\frac{1}{2}}|T_n^{(\mathbb{J})}|^p) \quad (2.32)$$

By definition of T_n and Cauchy's inequality, we have

$$\mathbf{E}^{\frac{1}{2}}|T_n^{(\mathbb{J})}|^p \leq \left(\frac{1}{n} \sum_{l \in \mathbb{T}_j} \mathbf{E}^{\frac{1}{2}}|\varepsilon_l^{(\mathbb{J})}|^{2p} \mathbf{E}^{\frac{1}{2}}|R_{ll}^{(\mathbb{J})}|^{2p} \right)^{\frac{1}{2}}. \quad (2.33)$$

Applying the inequality $|\varepsilon_l^{(\mathbb{J})}| \leq |\varepsilon_{l1}^{(\mathbb{J})}| + |\varepsilon_{l2}^{(\mathbb{J})}| + |\varepsilon_{l3}^{(\mathbb{J})}| + |\varepsilon_{l4}^{(\mathbb{J})}|$ and Lemmas 2.1, 2.4, 2.2 and [1, Lemma 7.12], we get

$$\mathbf{E}^{\frac{1}{2}}|T_n^{(\mathbb{J})}|^p \leq C^p (sH_0)^{\frac{p}{2}} \left(\left(\frac{p^4 s^2 H_0^2}{(nv)^{2(1-2\alpha)}} \right)^{\frac{p}{2}} + \left(\frac{C}{n^{(1-2\alpha)}} \right)^{\frac{p}{2}} \right). \quad (2.34)$$

We use here that $1 \leq 4v^{-1}$ and $p \geq 2p(1-2\alpha)$. Similarly we get

$$\mathbf{E}^{\frac{1}{2}}|\varepsilon_j^{(\mathbb{J})}|^{2p} \leq \left(\frac{p^4 s^2 H_0^2}{(nv)^{2(1-2\alpha)}} \right)^p + \left(\frac{C}{n^{(1-2\alpha)}} \right)^p. \quad (2.35)$$

Combining inequalities (2.32) – (2.35), we arrive at

$$\begin{aligned} \mathbf{E}|R_{jj}^{(\mathbb{J})}|^p &\leq 3^p + \left(\left(\frac{9C^2 p^4 (sH_0)^5}{(nv)^{2(1-2\alpha)}} \right)^{\frac{p}{2}} + \left(\frac{9C^2 s^3 H_0^3}{n^{(1-2\alpha)}} \right)^{\frac{p}{2}} + \left(\frac{3C p^4 (sH_0)^3}{(nv)^{2(1-2\alpha)}} \right)^p \right. \\ &\quad \left. + \left(\frac{3C s H_0}{n^{1-2\alpha}} \right)^p \right). \end{aligned} \quad (2.36)$$

This inequality ensures that we may choose the constants A_1 and A_0 such that, for $v \geq v_1/s$

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \leq H_0^p. \quad (2.37)$$

Thus Lemma 2.5 is proved. \square

To prove Theorem 2.1 it is enough to repeat the proof of Lemma 5.13 and Corollary 5.14 in [1].

3 Proof of Theorem 1.4

We return now to the representation (2.26) which implies that

$$s_n(z) = \frac{1}{n} \sum_{j=1}^n \mathbf{E} R_{jj} = s(z) + \mathbf{E} \Lambda_n = s(z) + \mathbf{E} \frac{T_n(z)}{z + s(z) + m_n(z)}. \quad (3.1)$$

The last equality may be further reformulated as

$$s_n(z) = s(z) + \mathbf{E} \frac{\frac{1}{n} \sum_{j=1}^n \varepsilon_{j4} R_{jj}}{z + s(z) + m_n(z)} + \mathbf{E} \frac{\widehat{T}_n(z)}{z + s(z) + m_n(z)}, \quad (3.2)$$

where

$$\widehat{T}_n = \sum_{\nu=1}^3 \frac{1}{n} \sum_{j=1}^n \varepsilon_{j\nu} R_{jj}.$$

Note that the definition of ε_{j4} in (2.26) ($\mathbb{J} = \emptyset$) and equality

$$\mathrm{Tr} \mathbf{R} - \mathrm{Tr} \mathbf{R}^{(j)} = (1 + \frac{1}{n} \sum_{l,k \in \mathbb{T}_j} X_{jl} X_{jk} [(R^{(j)})^2]_{kl}) R_{jj} = R_{jj}^{-1} \frac{dR_{jj}}{dz}, \quad (3.3)$$

(see as well [1, equality (7.34)]) together imply

$$\frac{1}{n} \sum_{j=1}^n \varepsilon_{j4} R_{jj} = \frac{1}{n} \mathrm{Tr} \mathbf{R}^2 = \frac{1}{n} \frac{dm_n(z)}{dz}. \quad (3.4)$$

Thus we may rewrite (3.2) as

$$s_n(z) = s(z) + \frac{1}{n} \mathbf{E} \frac{m'_n(z)}{z + s(z) + m_n(z)} + \mathbf{E} \frac{\widehat{T}_n(z)}{z + s(z) + m_n(z)}. \quad (3.5)$$

Denote

$$\mathfrak{T} = \mathbf{E} \frac{\widehat{T}_n(z)}{z + s(z) + m_n(z)}. \quad (3.6)$$

3.1 Estimation of \mathfrak{T}

We represent \mathfrak{T} as follows

$$\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2,$$

where

$$\begin{aligned}\mathfrak{T}_1 &= -\frac{1}{n} \sum_{j=1}^n \sum_{\nu=1}^3 \mathbf{E} \frac{\varepsilon_{j\nu} \frac{1}{z+m_n^{(j)}(z)}}{z+m_n(z)+s(z)}, \\ \mathfrak{T}_2 &= \frac{1}{n} \sum_{j=1}^n \sum_{\nu=1}^3 \mathbf{E} \frac{\varepsilon_{j\nu} (R_{jj} + \frac{1}{z+m_n^{(j)}(z)})}{z+m_n(z)+s(z)}.\end{aligned}$$

3.1.1 Estimation of \mathfrak{T}_1

We may decompose \mathfrak{T}_1 as

$$\mathfrak{T}_1 = \mathfrak{T}_{11} + \mathfrak{T}_{12}, \quad (3.7)$$

where

$$\begin{aligned}\mathfrak{T}_{11} &= -\frac{1}{n} \sum_{j=1}^n \sum_{\nu=1}^3 \mathbf{E} \frac{\varepsilon_{j\nu} \frac{1}{z+m_n^{(j)}(z)}}{z+m_n^{(j)}(z)+s(z)}, \\ \mathfrak{T}_{12} &= -\frac{1}{n} \sum_{j=1}^n \sum_{\nu=1}^3 \mathbf{E} \frac{\varepsilon_{j\nu} \varepsilon_{j4} \frac{1}{z+m_n^{(j)}(z)}}{(z+m_n^{(j)}(z)+s(z))(z+m_n(z)+s(z))}.\end{aligned}$$

It is easy to see that, by conditional expectation

$$\mathfrak{T}_{11} = 0. \quad (3.8)$$

Applying the Cauchy–Schwartz inequality, for $\nu = 1, 2, 3$, we get

$$\begin{aligned}& \left| \mathbf{E} \frac{\varepsilon_{j\nu} \varepsilon_{j4} \frac{1}{z+m_n^{(j)}(z)}}{(z+m_n^{(j)}(z)+s(z))(z+m_n(z)+s(z))} \right| \\ & \leq \mathbf{E}^{\frac{1}{2}} \left| \frac{\varepsilon_{j\nu}}{(z+m_n^{(j)}(z))(z+m_n^{(j)}(z)+s(z))} \right|^2 \mathbf{E}^{\frac{1}{2}} \left| \frac{\varepsilon_{j4}}{z+m_n(z)+s(z)} \right|^2. \quad (3.9)\end{aligned}$$

Conditioning and using [1, Lemmas 7.15, 7.16] together with the bound $\operatorname{Im} m_n^{(j)}(z) \leq |z+m_n^{(j)}(z)+s(z)|$, we get, for $\nu = 2, 3$,

$$\mathbf{E}^{\frac{1}{2}} \left| \frac{\varepsilon_{j\nu}}{(z+m_n^{(j)}(z))(z+m_n^{(j)}(z)+s(z))} \right|^2 \leq \frac{C}{\sqrt{nv}} \mathbf{E}^{\frac{1}{2}} \frac{1}{|z+m_n^{(j)}(z)|^2 |z+m_n^{(j)}(z)+s(z)|}.$$

Furthermore, applying [1, Lemma 7.5], we get

$$\mathbf{E}^{\frac{1}{2}} \left| \frac{\varepsilon_{j\nu}}{(z + m_n^{(j)}(z))(z + m_n^{(j)}(z) + s(z))} \right|^2 \leq \frac{C}{\sqrt{nv}|z^2 - 4|^{\frac{1}{4}}} \mathbf{E}^{\frac{1}{2}} \frac{1}{|z + m_n^{(j)}(z)|^2}. \quad (3.10)$$

Inequalities (3.9), (3.10), Theorem 2.1 together imply, for $\nu = 2, 3$,

$$\left| \mathbf{E} \frac{\varepsilon_{j\nu} \varepsilon_{j4} \frac{1}{z + m_n^{(j)}(z)}}{(z + m_n^{(j)}(z) + s(z))(z + m_n(z) + s(z))} \right| \leq \frac{C}{\sqrt{nv}|z^2 - 4|^{\frac{1}{4}}} \mathbf{E}^{\frac{1}{2}} \frac{|\varepsilon_{j4}|^2}{|z + m_n^{(j)}(z) + s(z)|^2}. \quad (3.11)$$

By [1, Lemma 7.5] we have for $\nu = 1$

$$\begin{aligned} \mathbf{E}^{\frac{1}{2}} \frac{|\varepsilon_{j\nu}|^2}{|z + m_n^{(j)}(z) + s(z)|^2 |z + m_n^{(j)}(z)|^2} &\leq \frac{C}{\sqrt{n}\sqrt{|z^2 - 4|}} \mathbf{E}^{\frac{1}{2}} \frac{1}{|z + m_n^{(j)}(z)|^2} \\ &\leq \frac{C}{\sqrt{nv}|z^2 - 4|^{\frac{1}{4}}} \mathbf{E}^{\frac{1}{2}} \frac{1}{|z + m_n^{(j)}(z)|^2}. \end{aligned} \quad (3.12)$$

Applying Theorem 2.1, we get

$$\mathbf{E}^{\frac{1}{2}} \frac{|\varepsilon_{j\nu}|^2}{|z + m_n^{(j)}(z) + s(z)|^2 |z + m_n^{(j)}(z)|^2} \leq \frac{C}{\sqrt{nv}|z^2 - 4|^{\frac{1}{4}}}. \quad (3.13)$$

Furthermore, according to Lemma 4.2, we have, for $z \in \mathbb{G}$,

$$\mathbf{E}^{\frac{1}{2}} \left| \frac{\varepsilon_{j4}}{z + m_n(z) + s(z)} \right|^2 \leq \frac{C}{nv}. \quad (3.14)$$

Finally we get

$$|\mathcal{T}_{12}| \leq \frac{C}{(nv)^{\frac{3}{2}}|z^2 - 4|^{\frac{1}{4}}}. \quad (3.15)$$

3.1.2 Estimation of \mathfrak{T}_2

Using the representation (2.26), we write

$$\mathfrak{T}_2 = \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\tilde{\varepsilon}_j^2 R_{jj}}{(z + m^{(j)}(z))(z + s(z) + m_n(z))}.$$

Furthermore we note that

$$\tilde{\varepsilon}_j^2 = (\varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3})^2 = \varepsilon_{j2}^2 + \eta_j,$$

where

$$\eta_j = (\varepsilon_{j1} + \varepsilon_{j3})^2 + 2(\varepsilon_{j1} + \varepsilon_{j3})\varepsilon_{j2}.$$

We now decompose \mathfrak{T}_2 as follows

$$\mathfrak{T}_2 = \mathfrak{T}_{21} + \mathfrak{T}_{22} + \mathfrak{T}_{23}, \quad (3.16)$$

where

$$\begin{aligned} \mathfrak{T}_{21} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2 R_{jj}}{(z + m^{(j)}(z))(z + s(z) + m_n(z))}, \\ \mathfrak{T}_{22} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\eta_j R_{jj}}{(z + m^{(j)}(z))(z + s(z) + m_n^{(j)}(z))}, \\ \mathfrak{T}_{23} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\eta_j R_{jj} \varepsilon_{j4}}{(z + m^{(j)}(z))(z + s(z) + m_n^{(j)}(z))(z + s(z) + m_n(z))}. \end{aligned}$$

First we note that

$$|\eta_j| \leq 2|\varepsilon_{j1}|^2 + 2|\varepsilon_{j3}|^2 + 2|\varepsilon_{j2}|(|\varepsilon_{j1}| + |\varepsilon_{j2}|). \quad (3.17)$$

Applying the Cauchy – Schwartz inequality, we obtain

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{(2|\varepsilon_{j1}|^2 + 2|\varepsilon_{j3}|^2)|R_{jj}|}{|z + m^{(j)}(z)||z + s(z) + m_n^{(j)}(z)|} &\leq \frac{C}{n} \sum_{j=1}^n \mathbf{E}^{\frac{4}{4+\kappa}} \frac{|\varepsilon_{j1}|^{\frac{4+\kappa}{2}} + |\varepsilon_{j3}|^{\frac{4+\kappa}{2}}}{|z + s(z) + m_n^{(j)}(z)|^{\frac{4+\kappa}{4}}} \\ &\quad \times \mathbf{E}^{\frac{\kappa}{(4+\kappa)}} |R_{jj}|^{\frac{4+\kappa}{\kappa}} |z + m^{(j)}(z)|^{-\frac{(4+\kappa)}{\kappa}}. \end{aligned} \quad (3.18)$$

Lemma 4.3 and [1, Lemma 7.5 inequality (7.9)] together imply

$$\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{(2|\varepsilon_{j1}|^2 + 2|\varepsilon_{j3}|^2)|R_{jj}|}{|z + m^{(j)}(z)||z + s(z) + m_n^{(j)}(z)|} \leq \frac{C}{n|z^2 - 4|^{\frac{1}{2}}}. \quad (3.19)$$

Furthermore, applying Hölder's inequality, we get

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{2(|\varepsilon_{j1}| + |\varepsilon_{j3}|)|\varepsilon_{j2}||R_{jj}|}{|z + m^{(j)}(z)||z + s(z) + m_n^{(j)}(z)|} &\leq \frac{C}{n} \sum_{j=1}^n \mathbf{E}^{\frac{2}{4+\kappa}} \frac{|\varepsilon_{j1}|^{\frac{4+\kappa}{2}} + |\varepsilon_{j3}|^{\frac{4+\kappa}{2}}}{|z + s(z) + m_n^{(j)}(z)|^{\frac{4+\kappa}{4}}} \\ &\quad \times \mathbf{E}^{\frac{2}{4+\kappa}} \frac{|\varepsilon_{j2}|^{\frac{4+\kappa}{2}}}{|z + s(z) + m_n^{(j)}(z)|^{\frac{4+\kappa}{4}}} \mathbf{E}^{\frac{\kappa}{(4+\kappa)}} |R_{jj}|^{\frac{4+\kappa}{\kappa}} |z + m^{(j)}(z)|^{-\frac{(4+\kappa)}{\kappa}}. \end{aligned} \quad (3.20)$$

Applying Lemmas 4.4, 4.3 and [1, Lemma 7.5 inequality (7.9)], we obtain

$$\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{(2(|\varepsilon_{j1}| + |\varepsilon_{j3}|)|\varepsilon_{j2}|)|R_{jj}|}{|z + m^{(j)}(z)||z + s(z) + m_n^{(j)}(z)|} \leq \frac{C}{n\sqrt{v}|z^2 - 4|^{\frac{1}{4}}}. \quad (3.21)$$

Combining inequalities (3.19) and (3.21), we get

$$\mathcal{T}_{22} \leq \frac{C}{nv^{\frac{3}{4}}}. \quad (3.22)$$

Applying [1, inequality (7.42)], we obtain

$$|\mathfrak{T}_{23}| \leq \frac{C}{n} \sum_{j=1}^n \mathbf{E} \frac{|\eta_j||R_{jj}|}{|z + m^{(j)}(z)||z + s(z) + m_n^{(j)}(z)|} \quad (3.23)$$

Repeating the arguments of inequality (3.22), we similarly get

$$|\mathfrak{T}_{23}| \leq \frac{C}{nv^{\frac{3}{4}}}. \quad (3.24)$$

We continue now with \mathfrak{T}_{21} . We represent it in the form

$$\mathfrak{T}_{21} = H_1 + H_2, \quad (3.25)$$

where

$$H_1 = -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2}{(z + m^{(j)}(z))^2(z + s(z) + m_n(z))},$$

$$H_2 = \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2(R_{jj} + \frac{1}{z + m_n^{(j)}(z)})}{(z + m^{(j)}(z))(z + s(z) + m_n(z))}.$$

Furthermore, using the representation

$$R_{jj} = -\frac{1}{z + m_n^{(j)}(z)} + \frac{1}{z + m_n^{(j)}(z)}(\varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3})R_{jj}, \quad (3.26)$$

we bound H_2 in the following way

$$|H_2| \leq H_{21} + H_{22} + H_{23},$$

where

$$\begin{aligned} H_{21} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{4|\varepsilon_{j1}|^3 |R_{jj}|}{|z + m^{(j)}(z)|^2 |z + s(z) + m_n(z)|}, \\ H_{22} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{2|\varepsilon_{j2}|^3 |R_{jj}|}{|z + m^{(j)}(z)|^2 |z + s(z) + m_n(z)|}, \\ H_{23} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{2|\varepsilon_{j3}| |\varepsilon_{j2}|^2 |R_{jj}|}{|z + m^{(j)}(z)|^2 |z + s(z) + m_n(z)|}. \end{aligned}$$

Using [1, inequality (7.42)], we get, for $\nu = 1, 2$

$$\begin{aligned} H_{2\nu} &\leq \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{4|\varepsilon_{j\nu}|^3 |R_{jj}|}{|z + m^{(j)}(z)|^2 |z + s(z) + m_n^{(j)}(z)|} \\ &\quad + \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{4|\varepsilon_{j\nu}|^3 |R_{jj}| |\varepsilon_{j4}|}{|z + m^{(j)}(z)|^2 |z + s(z) + m_n^{(j)}(z)| |z + s(z) + m_n(z)|} \\ &\leq \frac{C}{n} \sum_{j=1}^n \mathbf{E}^{\frac{3}{4}} \frac{|\varepsilon_{j\nu}|^4}{|z + m_n^{(j)}(z)|^{\frac{8}{3}} |z + s(z) + m_n^{(j)}(z)|^{\frac{4}{3}}} \mathbf{E}^{\frac{1}{4}} |R_{jj}|^4. \end{aligned} \quad (3.27)$$

Applying [1, Corollary 7.17 inequality (7.32)] with $\beta = \frac{4}{3}$ and $\alpha = \frac{8}{3}$, we obtain, for $z \in \mathbb{G}$, and for $\nu = 1, 2$

$$\mathbf{E}^{\frac{3}{4}} \frac{|\varepsilon_{j\nu}|^4}{|z + m^{(j)}(z)|^{\frac{8}{3}} |z + s(z) + m_n^{(j)}(z)|^{\frac{4}{3}}} \leq \frac{C}{(nv)^{\frac{3}{2}}}.$$

Furthermore, using Hölder's inequality, we get

$$H_{23} \leq \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{2}{4+\varkappa}} \frac{|\varepsilon_{j2}|^{4+\varkappa}}{|z + s(z) + m_n^{(j)}(z)|^{\frac{4+\varkappa}{4}}} \mathbf{E}^{\frac{2}{4+\varkappa}} |\varepsilon_{j3}|^{\frac{4+\varkappa}{2}} \mathbf{E}^{\frac{\varkappa}{4+\varkappa}} \left(\frac{|R_{jj}|}{|z + m^{(j)}(z)|} \right)^{\frac{4+\varkappa}{\varkappa}}. \quad (3.28)$$

Applying Lemmas 4.4, 4.3, we get

$$H_{23} \leq \frac{C}{n^{\frac{3}{2}} v} \leq \frac{C}{(nv)^{\frac{3}{2}}}. \quad (3.29)$$

Combining inequalities (3.28) and (3.29), we arrive at

$$H_2 \leq \frac{C}{(nv)^{\frac{3}{2}}}. \quad (3.30)$$

Consider now H_1 . Using the equality

$$\frac{1}{z + m_n(z) + s(z)} = \frac{1}{z + 2s(z)} - \frac{\Lambda_n(z)}{(z + 2s(z))(z + m_n(z) + s(z))}$$

and

$$\Lambda_n = \Lambda_n^{(j)} + \varepsilon_{j4}, \quad (3.31)$$

we represent it in the form

$$H_1 = H_{11} + H_{12} + H_{13}, \quad (3.32)$$

where

$$\begin{aligned} H_{11} &= -\frac{1}{(z + s(z))^2} \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2}{z + s(z) + m_n(z)} \\ &= -s^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2}{z + s(z) + m_n(z)}, \\ H_{12} &= -\frac{1}{(z + s(z))} \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2 \Lambda_n^{(j)}}{(z + m_n^{(j)}(z))^2 (z + s(z) + m_n(z))}, \\ H_{13} &= -\frac{1}{(z + s(z))^2} \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2 \Lambda_n^{(j)}}{(z + m_n^{(j)}(z))(z + s(z) + m_n(z))}. \end{aligned}$$

In order to apply conditional independence, we write

$$H_{11} = H_{111} + H_{112},$$

where

$$\begin{aligned} H_{111} &= -s^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2}{z + m_n^{(j)}(z) + s(z)}, \\ H_{112} &= s^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2 \varepsilon_{j4}}{(z + s(z) + m_n(z))(z + m_n^{(j)}(z) + s(z))}. \end{aligned}$$

It is straightforward to check that

$$\mathbf{E}\{\varepsilon_{j2}^2 | \mathfrak{M}^{(j)}\} = \frac{1}{n^2} \text{Tr}(\mathbf{R}^{(j)})^2 - \frac{1}{n^2} \sum_{l \in \mathbb{T}_j} (R_{ll}^{(j)})^2.$$

Using equality (3.4) for $m'_n(z)$ and the corresponding relation for $m_n^{(j)'}(z)$, we may write

$$H_{111} = L_1 + L_2 + L_3 + L_4,$$

where

$$L_1 = -s^2(z) \frac{1}{n} \mathbf{E} \frac{m'_n(z)}{z + m_n(z) + s(z)},$$

$$L_2 = s^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\frac{1}{n^2} \sum_{l \in \mathbb{T}_j} (R_{il}^{(j)})^2}{z + m_n^{(j)}(z) + s(z)}, \quad (3.33)$$

$$L_3 = s^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\frac{1}{n} ((m_n^{(j)}(z))' - m'_n(z))}{z + m_n^{(j)}(z) + s(z)}, \quad (3.34)$$

$$L_4 = s^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\frac{1}{n} ((m_n^{(j)}(z))' - m'_n(z)) \varepsilon_{j4}}{(z + m_n(z) + s(z))(z + m_n^{(j)}(z) + s(z))}.$$

Using [1, Lemma 7.6 inequality (7.11)] and Theorem 2.1, we get

$$|L_2| \leq \frac{C}{n\sqrt{|z^2 - 4|}}. \quad (3.35)$$

Note that

$$m_n^{(j)}(z)' - m'_n(z) = \int_{-\infty}^{\infty} \frac{1}{(x - z)^2} d(\mathcal{F}_n^{(j)}(x) - \mathcal{F}_n(x)). \quad (3.36)$$

Integrating by parts we obtain

$$|m_n^{(j)}(z)' - m'_n(z)| \leq \frac{C}{nv^2}. \quad (3.37)$$

The last inequality and [1, Lemma 7.5 inequality (7.9)] together imply

$$|L_3| \leq \frac{C}{n^2 v^2 \sqrt{|z^2 - 4|}}. \quad (3.38)$$

Finally, using that $|\varepsilon_{j4}|/|z + s(z) + m_n(z)| \leq C$ for $z \in \mathbb{G}$, we arrive at

$$|L_4| \leq \frac{C}{n^2 v^2 \sqrt{|z^2 - 4|}}. \quad (3.39)$$

Conditioning on $\mathfrak{M}^{(j)}$ and applying Lemma 4.4, we get

$$|H_{112}| \leq \frac{C}{n^2 v^2 |z^2 - 4|^{\frac{1}{2}}}. \quad (3.40)$$

Applying [1, inequality (7.42)] , we may write

$$|H_{12}| \leq \frac{C}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j2}|^2 |\Lambda_n^{(j)}|}{|z + m_n^{(j)}(z)| |z + m_n^{(j)}(z) + s(z)|}.$$

Conditioning on $\mathfrak{M}^{(j)}$ and applying Lemma 4.4, we get

$$|H_{12}| \leq \frac{C}{nv} \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{|\Lambda_n^{(j)}|}{|z + m_n^{(j)}(z)|} \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{n^2 v^2}.$$

By Lemma 4.5, we get

$$|H_{12}| \leq \frac{C}{n^2 v^2}. \quad (3.41)$$

Similarly we obtain

$$|H_{13}| \leq \frac{C}{n^2 v^2}. \quad (3.42)$$

Now we rewrite the equations (3.2) and (3.5) as follows, using the remainder term \mathfrak{T}_3 , which can be bounded by means of inequalities (3.41), (3.22), (3.35), (3.38), (3.39), (3.40) and (3.42)

$$\mathbf{E} \Lambda_n(z) = \mathbf{E} m_n(z) - s(z) = \frac{(1 - s^2(z))}{n} \mathbf{E} \frac{m'_n(z)}{z + m_n(z) + s(z)} + \mathfrak{T}_3, \quad (3.43)$$

where

$$|\mathfrak{T}_3| \leq \frac{C}{nv^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2 - 4|^{\frac{1}{4}}}.$$

Note that

$$1 - s^2(z) = s(z) \sqrt{z^2 - 4}.$$

In (3.43) it remains to estimate the quantity

$$\mathfrak{T}_4 = -\frac{s(z) \sqrt{z^2 - 4}}{n} \mathbf{E} \frac{m'_n(z)}{z + m_n(z) + s(z)}.$$

3.2 Estimation of \mathfrak{T}_4

Using that $\Lambda_n = m_n(z) - s(z)$ we rewrite \mathfrak{T}_4 as

$$\mathfrak{T}_4 = \mathfrak{T}_{41} + \mathfrak{T}_{42} + \mathfrak{T}_{43},$$

where

$$\begin{aligned}\mathfrak{T}_{41} &= -\frac{s(z)s'(z)}{n}, \\ \mathfrak{T}_{42} &= \frac{s(z)\sqrt{z^2-4}}{n} \mathbf{E} \frac{m'_n(z) - s'(z)}{z + m_n(z) + s(z)}, \\ \mathfrak{T}_{43} &= \frac{s(z)}{n} \mathbf{E} \frac{(m'_n(z) - s'(z))\Lambda_n}{z + m_n(z) + s(z)}.\end{aligned}$$

3.2.1 Estimation of \mathfrak{T}_{42}

First we investigate $m'_n(z)$. The following equality holds

$$m'_n(z) = \frac{1}{n} \text{Tr } R^2 = \sum_{j=1}^n \varepsilon_{j4} R_{jj} = s^2(z) \sum_{j=1}^n \varepsilon_{j4} R_{jj}^{-1} + D_1, \quad (3.44)$$

where

$$D_1 = \sum_{j=1}^n \varepsilon_{j4} (R_{jj} - s(z))(1 + R_{jj}^{-1} s(z)). \quad (3.45)$$

Using equality (3.3), we may write

$$m'_n(z) = \frac{s^2(z)}{n} \sum_{j=1}^n \left(1 + \frac{1}{n} \sum_{l,k \in \mathbb{T}_j} X_{jl} X_{jk} [(R^{(j)})^2]_{lk}\right) + D_1.$$

Denote

$$\begin{aligned}\beta_{j1} &= \frac{1}{n} \sum_{l \in \mathbb{T}_j} [(R^{(j)})^2]_{ll} - \frac{1}{n} \sum_{l=1}^n [(R)^2]_{ll} = \frac{1}{n} \sum_{l \in \mathbb{T}_j} [(R^{(j)})^2]_{ll} - m'_n(z) \\ &= \frac{1}{n} \frac{d}{dz} (\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)}), \\ \beta_{j2} &= \frac{1}{n} \sum_{l \in \mathbb{T}_j} (X_{jl}^2 - 1) [(R^{(j)})^2]_{ll}, \\ \beta_{j3} &= \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} X_{jl} X_{jk} [(R^{(j)})^2]_{lk}.\end{aligned} \quad (3.46)$$

Using these notations we may write

$$m'_n(z) = s^2(z)(1 + m'_n(z)) + \frac{s^2(z)}{n} \sum_{j=1}^n (\beta_{j1} + \beta_{j2} + \beta_{j3}) + D_1.$$

Solving this equation with respect to $m'_n(z)$ we obtain

$$m'_n(z) = \frac{s^2(z)}{1 - s^2(z)} + \frac{1}{1 - s^2(z)}(D_1 + D_2), \quad (3.47)$$

where

$$D_2 = \frac{s^2(z)}{n} \sum_{j=1}^n (\beta_{j1} + \beta_{j2} + \beta_{j3}).$$

Note that for the semi-circular law the following identities hold

$$\frac{s^2(z)}{1 - s^2(z)} = \frac{s^2(z)}{1 + \frac{s(z)}{z+s(z)}} = -\frac{s(z)}{z + 2s(z)} = s'(z).$$

Applying this relation we rewrite equality (3.47) as

$$m'_n(z) - s'(z) = \frac{1}{s(z)(z + 2s(z))}(D_1 + D_2). \quad (3.48)$$

Using the last equality, we may represent \mathfrak{T}_{42} now as follows

$$\mathfrak{T}_{42} = \mathfrak{T}_{421} + \mathfrak{T}_{422},$$

where

$$\begin{aligned} \mathfrak{T}_{421} &= \frac{1}{n} \mathbf{E} \frac{D_1}{z + m_n(z) + s(z)}, \\ \mathfrak{T}_{422} &= \frac{1}{n} \mathbf{E} \frac{D_2}{z + m_n(z) + s(z)}. \end{aligned}$$

Recall that, by (3.45),

$$\mathfrak{T}_{421} = -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j4}(R_{jj} - s(z))(1 + R_{jj}^{-1}s(z))}{(z + s(z) + m_n(z))}. \quad (3.49)$$

Using that $|\varepsilon_{j4}| \leq 1/nv$ and

$$R_{jj} - s(z) = s(z)(\varepsilon_j + \Lambda_n)R_{jj} \quad (3.50)$$

and that $|z + m_n(z) + s(z)| \geq c|z^2 - 4|^{\frac{1}{2}}$ for $z \in \mathbb{G}$, it is straightforward to check that

$$|\mathfrak{T}_{421}| \leq \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|z^2 - 4|^{\frac{1}{4}}}. \quad (3.51)$$

3.2.2 Estimation of \mathfrak{T}_{422}

We represent now \mathfrak{T}_{422} in the form

$$\mathfrak{T}_{422} = \mathfrak{T}_{51} + \mathfrak{T}_{52} + \mathfrak{T}_{53},$$

where

$$\mathfrak{T}_{5\nu} = \frac{1}{n^2} \sum_{j=1}^n \mathbf{E} \frac{\beta_{j\nu}}{z + m_n(z) + s(z)}, \quad \text{for } \nu = 1, 2, 3.$$

At first we investigate \mathfrak{T}_{53} . Note that, by [1, Lemma 7.26],

$$|\beta_{j1}| \leq \frac{C}{nv^2}.$$

Therefore, for $z \in \mathbb{G}$, using [1, Lemma 7.5 inequality (7.9)],

$$|\mathfrak{T}_{51}| \leq \frac{C}{n^2 v^2 \sqrt{|z^2 - 4|}} \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2 - 4|^{\frac{1}{4}}}. \quad (3.52)$$

Furthermore, we consider the quantity $\mathfrak{T}_{5\nu}$, for $\nu = 2, 3$. Applying the Cauchy-Schwartz inequality and [1, inequality (7.42)] as well, we get

$$|\mathfrak{T}_{5\nu}| \leq \frac{C}{n^2} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} \frac{|\beta_{j\nu}|^2}{|z + m_n^{(j)}(z) + s(z)|^2}.$$

By [1, Lemma 7.25] together with [1, Lemma 7.5], we obtain

$$\mathbf{E}^{\frac{1}{2}} \frac{|\beta_{j\nu}|^2}{|z + m_n^{(j)}(z) + s(z)|^2} \leq \frac{C}{n^{\frac{1}{2}} v^{\frac{3}{2}} |z^2 - 4|^{\frac{1}{4}}}.$$

This implies that

$$|\mathfrak{T}_{5\nu}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2 - 4|^{\frac{1}{4}}}. \quad (3.53)$$

Inequalities (3.52) and (3.53) yield

$$|\mathfrak{T}_{422}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2 - 4|^{\frac{1}{4}}}.$$

Combining (3.51) and (3.54), we get, for $z \in \mathbb{G}$,

$$|\mathfrak{T}_{42}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2 - 4|^{\frac{1}{4}}}. \quad (3.54)$$

3.2.3 Estimation of \mathfrak{T}_{43}

Recall that

$$\mathfrak{T}_{43} = \frac{s(z)}{n} \mathbf{E} \frac{(m'_n(z) - s'(z))\Lambda_n}{z + m_n(z) + s(z)}.$$

Applying equality (3.48), we obtain

$$\mathfrak{T}_{43} = \mathfrak{T}_{431} + \mathfrak{T}_{432},$$

where

$$\begin{aligned} \mathfrak{T}_{431} &= \frac{1}{n(z + 2s(z))} \mathbf{E} \frac{D_1 \Lambda_n}{z + m_n(z) + s(z)}, \\ \mathfrak{T}_{432} &= \frac{1}{n(z + 2s(z))} \mathbf{E} \frac{D_2 \Lambda_n}{z + m_n(z) + s(z)}. \end{aligned} \quad (3.55)$$

By definition of D_1 , we get

$$|\mathfrak{T}_{431}| \leq \mathfrak{T}_{61} + \mathfrak{T}_{62},$$

where

$$\begin{aligned} \mathfrak{T}_{61} &= \frac{C}{nv|z^2 - 4|^{\frac{1}{2}}} \frac{1}{n} \sum_{j=1}^n \mathbf{E} |\varepsilon_j| (|R_{jj}| + 1) |\Lambda_n|, \\ \mathfrak{T}_{62} &= \frac{C}{nv} \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{(|R_{jj}| + 1) |\Lambda_n|^2}{|z + m_n(z) + s(z)|}. \end{aligned} \quad (3.56)$$

Applying the Cauchy-Schwartz inequality and Lemma 4.5, we get

$$\mathfrak{T}_{61} \leq \frac{C}{n^2 v^2 |z^2 - 4|^{\frac{1}{2}}}. \quad (3.57)$$

Furthermore, using that $|\lambda_n| \leq \sqrt{|T_n|}$ for $z \in \mathbb{G}$, we get

$$\mathfrak{T}_{62} \leq \frac{C}{nv} \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} (|R_{jj}|^2 + 1) \mathbf{E}^{\frac{1}{2}} \frac{|T_n|^2}{|z + m_n(z) + s(z)|^2}. \quad (3.58)$$

By definition of T_n and Hölder's inequality, we have

$$\mathbf{E} \frac{|T_n|^2}{|z + m_n(z) + s(z)|^2} \leq \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{4}{4+\varkappa}} \frac{|\varepsilon_j|^{\frac{4+\varkappa}{2}}}{|z + m_n(z) + s(z)|^{\frac{4+\varkappa}{2}}} \mathbf{E}^{\frac{\varkappa}{4+\varkappa}} |R_{jj}|^{\frac{2(4+\varkappa)}{\varkappa}}. \quad (3.59)$$

Applying now Lemmas 4.4, 4.3 and Theorem 2.1, we get

$$\mathbf{E} \frac{|T_n|^2}{|z + m_n(z) + s(z)|^2} \leq \frac{C}{n|z^2 - 4|} + \frac{C}{nv|z^2 - 4|^{\frac{1}{2}}}. \quad (3.60)$$

The last inequality together with inequality (3.58) implies, for $z \in \mathbb{G}$,

$$\mathfrak{T}_{62} \leq \frac{C}{(nv)^{\frac{3}{2}}|z^2 - 4|^{\frac{1}{4}}}. \quad (3.61)$$

For $z \in \mathbb{G}$ we get

$$|\mathfrak{T}_{431}| \leq \frac{4}{n^{\frac{3}{2}}v^{\frac{3}{2}}|z^2 - 4|^{\frac{1}{4}}}.$$

Applying the Cauchy – Schwartz inequality, we get for \mathfrak{T}_{432} accordingly

$$|\mathfrak{T}_{432}| \leq \frac{C}{n|z^2 - 4|^{\frac{1}{2}}} \mathbf{E}^{\frac{1}{2}} |D_2|^2 \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2.$$

By Lemma 4.5, we have

$$|\mathfrak{T}_{432}| \leq \frac{C}{n^2v|z^2 - 4|^{\frac{1}{2}}} \mathbf{E}^{\frac{1}{2}} |D_2|^2. \quad (3.62)$$

By definition of D_2 ,

$$\mathbf{E} |D_2|^2 \leq \frac{1}{n} \sum_{j=1}^n (\mathbf{E} |\beta_{j1}|^2 + \mathbf{E} |\beta_{j2}|^2 + \mathbf{E} |\beta_{j3}|^2).$$

Applying [1, Lemma 7.26], [1, Lemma 7.25] with $\nu = 2, 3$, we get

$$\mathbf{E} |D_2|^2 \leq \frac{C}{n^2v^4} + \frac{C}{nv^3}. \quad (3.63)$$

Inequalities (3.62) and (3.63) together imply, for $z \in \mathbb{G}$,

$$|\mathfrak{T}_{432}| \leq \frac{C}{n^3v^3|z^2 - 4|^{\frac{1}{2}}} + \frac{C}{n^{\frac{5}{2}}v^{\frac{5}{2}}|z^2 - 4|^{\frac{1}{2}}} \leq \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|z^2 - 4|^{\frac{1}{4}}}.$$

Finally we observe that

$$s'(z) = -\frac{s(z)}{\sqrt{z^2 - 4}}$$

and, therefore

$$|\mathfrak{T}_{41}| \leq \frac{C}{n|z^2 - 4|^{\frac{1}{2}}}.$$

For $z \in \mathbb{G}$ we may rewrite it

$$|\mathfrak{T}_{41}| \leq \frac{C}{n\sqrt{v}}. \quad (3.64)$$

Combining now relations (3.43), (3.32), (3.52), (3.54), (3.64), we get for $z \in \mathbb{G}$,

$$|\mathbf{E}\Lambda_n| \leq \frac{C}{nv^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|z^2 - 4|^{\frac{1}{4}}}.$$

The last inequality completes the proof of Theorem 1.4.

4 Appendix

We reformulate here a result by Gine, Latala and Zinn (2000), [2], for quadratic forms $Q := \sum_{1 \leq l \neq k \leq n} a_{lk} \xi_l \xi_k$.

Lemma 4.1. *Assume that for $q \geq 4$ and for any $j = 1, \dots, n$*

$$\mathbf{E}|\xi_j|^q \leq \mu_q.$$

Then there exists an absolute constant $K > 0$ such that, for $z \in \mathbb{G}$,

$$\mathbf{E}|Q|^q \leq K^p \left(q^q \sigma^q \left(\sum_{1 \leq l \neq k \leq n} |a_{lk}|^2 \right)^{\frac{q}{2}} + \mu_q q^{\frac{3q}{2}} \sum_l \left(\sum_k |a_{kl}|^2 \right)^{\frac{q}{2}} + q^{2q} \mu_q^2 \sum_{l,k} |a_{lk}|^q \right). \quad (4.1)$$

Proof. For the proof see [2, Section 3, inequality (3.3)] with $h_{ij} = a_{ij} \xi_i \xi_j$. \square

Lemma 4.2. *Assuming the conditions of Theorem 1.3, we have, for any $\theta > 0$*

$$\mathbf{E} \left| \frac{\varepsilon_{j4}}{z + s(z) + m_n(z)} \right|^2 \frac{1}{|z + m_n^{(j)}(z)|^\theta} \leq \frac{C}{n^2 v^2} \quad (4.2)$$

with some positive constant C depending on \varkappa , A_\varkappa , D and θ .

Proof. Applying [1, inequality (7.42)], we may write

$$\mathbf{E} \left| \frac{\varepsilon_{j4}}{z + s(z) + m_n(z)} \right|^2 \frac{1}{|z + m_n^{(j)}(z)|^\theta} \leq \mathbf{E} \left| \frac{\varepsilon_{j4}}{z + s(z) + m_n^{(j)}(z)} \right|^2 \frac{1}{|z + m_n^{(j)}(z)|^\theta}. \quad (4.3)$$

Furthermore, using [1, equality (7.34)], we get

$$\begin{aligned} & \mathbf{E} \left| \frac{\varepsilon_{j4}}{z + s(z) + m_n^{(j)}(z)} \right|^2 \frac{1}{|z + m_n^{(j)}(z)|^\theta} \\ &= \mathbf{E} \left| \frac{(1 + \frac{1}{n} \sum_{k \neq l \in \mathbb{T}_j} X_{jk} X_{jl} [\mathbf{R}^{(j)^2}]_{lk}) R_{jj}}{z + s(z) + m_n^{(j)}(z)} \right|^2 \frac{1}{|z + m_n^{(j)}(z)|^\theta}. \end{aligned} \quad (4.4)$$

Introduce notations as in [1]:

$$\begin{aligned} \eta_{j0} &:= \frac{1}{n} (1 + \frac{1}{n} \sum_{l \in \mathbb{T}_j} [\mathbf{R}^{(j)^2}]_{ll}), \quad \eta_{j1} := \frac{1}{n^2} \sum_{l \in \mathbb{T}_j} (X_{jl}^2 - 1) [\mathbf{R}^{(j)^2}]_{ll}, \\ \eta_{j2} &:= \frac{1}{n^2} \sum_{l \neq k \in \mathbb{T}_j} X_{jl} X_{jk} [\mathbf{R}^{(j)^2}]_{lk}. \end{aligned} \quad (4.5)$$

Using that

$$|\eta_{j0}| \leq \frac{1}{nv} \operatorname{Im}(z + m_n^{(j)}(z)) \leq \frac{1}{nv} \operatorname{Im}(z + m_n^{(j)}(z) + s(z)), \quad (4.6)$$

(see [1, inequality (7.57)]) we get

$$\mathbf{E} \left| \frac{\eta_{j0}}{z + s(z) + m_n^{(j)}(z)} \right|^2 \frac{1}{|z + m_n^{(j)}(z)|^\theta} \leq \frac{1}{(nv)^2} \mathbf{E} \frac{1}{|z + m_n^{(j)}(z)|^\theta} \leq \frac{C}{(nv)^2}. \quad (4.7)$$

Note that

$$\mathbf{E}\{|\eta_{j1}|^2 \mid \mathfrak{M}^{(j)}\} \leq \frac{1}{n^4} \sum_{l \in \mathbb{T}_j} |[\mathbf{R}^{(j)^2}]_{ll}^2| \leq \frac{C}{n^3 v^2} \quad (4.8)$$

and

$$\mathbf{E}\{|\eta_{j2}|^2 \mid \mathfrak{M}^{(j)}\} \leq \frac{1}{n^4} \sum_{l \neq k \in \mathbb{T}_j} |[\mathbf{R}^{(j)^2}]_{lk}^2| \leq \frac{C}{n^3 v^3} \operatorname{Im} m_n^{(j)}(z). \quad (4.9)$$

See [1, Lemma 6 inequalities (7.15), (7.16)]. Inequalities (4.8), (4.9) together imply

$$\begin{aligned} & \mathbf{E} \left| \frac{\eta_{j1} + \eta_{j2}}{z + s(z) + m_n(z)} \right|^2 \frac{1}{|z + m_n^{(j)}(z)|^\theta} \\ & \leq \frac{C}{n^3 v^3} \mathbf{E} \left| \frac{v + \operatorname{Im} m_n^{(j)}(z)}{z + s(z) + m_n(z)} \right|^2 \frac{1}{|z + m_n^{(j)}(z)|^\theta} \\ & \leq \frac{C}{n^3 v^3 |z^2 - 4|^{\frac{1}{2}}}. \end{aligned} \quad (4.10)$$

We use here [1, Lemma 7.5 inequality (7.9)]. Furthermore, applying [1, Lemma 7.13], we get, for $z \in \mathbb{G}$

$$\mathbf{E} \left| \frac{\eta_{j1} + \eta_{j2}}{z + s(z) + m_n(z)} \right|^2 \frac{1}{|z + m_n^{(j)}(z)|^\theta} \leq \frac{C}{n^2 v^2}. \quad (4.11)$$

Inequalities (4.2) and (4.8) conclude the proof of Lemma 4.2. Thus Lemma 4.2 is proved. \square

Lemma 4.3. *Assuming the conditions of Theorem 1.3 there exists a positive constant depending on $\varkappa, \mu_{4+\varkappa}$ and D such that*

$$\mathbf{E}^{\frac{2}{4+\varkappa}} \{ |\varepsilon_{j3}|^{\frac{4+\varkappa}{2}} |\mathfrak{M}^{(j)}\} \leq \frac{C}{\sqrt{n}} \left(\frac{1}{n} \sum_{l=1}^n |R_{ll}|^2 \right)^{\frac{1}{2}} \quad (4.12)$$

and

$$\mathbf{E}^{\frac{2}{4+\varkappa}} \{ |\varepsilon_{j3}|^{\frac{4+\varkappa}{2}} |\mathfrak{M}^{(j)}\} \leq \frac{C}{\sqrt{nv}} (\operatorname{Im} m_n^{(j)}(z))^{\frac{1}{2}} \quad (4.13)$$

Proof. Applying Rosenthal's inequality, we get

$$\mathbf{E} |\varepsilon_{j3}|^{\frac{4+\varkappa}{2}} \leq \frac{C\mu_4}{n^{\frac{4+\varkappa}{2}}} \left(\sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^2 \right)^{\frac{4+\varkappa}{4}} + \frac{C}{n^{\frac{4+\varkappa}{2}}} \mu_{4+\varkappa} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^{\frac{4+\varkappa}{2}}. \quad (4.14)$$

Note that

$$\sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^{\frac{4+\varkappa}{2}} \leq \left(\sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^2 \right)^{\frac{4+\varkappa}{4}}. \quad (4.15)$$

\square

These inequalities complete the proof of the first claim of Lemma 4.3. The second one is easy. Thus Lemma 4.3 is proved.

Lemma 4.4. *Assuming the conditions of Theorem 1.3 there exists a positive constant depending on $\varkappa, \mu_{4+\varkappa}$ and D such that*

$$\mathbf{E}^{\frac{1}{4+\varkappa}} \{ |\varepsilon_{j2}|^{4+\varkappa} |\mathfrak{M}^{(j)}\} \leq \frac{C}{\sqrt{nv}} \operatorname{Im}^{\frac{1}{2}} m_n^{(j)}(z). \quad (4.16)$$

Proof. Applying Lemma 4.1, we get

$$\begin{aligned} \mathbf{E} \{ |\varepsilon_{j2}|^{4+\varkappa} |\mathfrak{M}^{(j)}\} &\leq C n^{-(4+\varkappa)} \left(\left(\sum_{l \neq k \in \mathbb{T}_j} |R_{kl}^{(j)}|^2 \right)^{\frac{4+\varkappa}{2}} + \mu_{4+\varkappa} \sum_{l \in \mathbb{T}_j} \left(\sum_{k \in \mathbb{T}_{k,j}} |R_{kl}^{(j)}|^2 \right)^{\frac{4+\varkappa}{2}} \right. \\ &\quad \left. + \mu_{4+\varkappa}^2 \sum_{l \in \mathbb{T}_j} \sum_{k \in \mathbb{T}_{k,j}} |R_{kl}^{(j)}|^{4+\varkappa} \right) \leq C n^{-(4+\varkappa)} \left(\sum_{k \in \mathbb{T}_{k,j}} |R_{kl}^{(j)}|^2 \right)^{\frac{4+\varkappa}{2}} \end{aligned}$$

Now [1, Lemma 7.6 inequality (7.11)] completes the proof. Thus Lemma 4.4 is proved. \square

Lemma 4.5. *Assuming the conditions of Theorem 1.3 there exist a positive constant depending on \varkappa , D such that for any $z \in \mathbb{G}$*

$$\mathbf{E}|\Lambda_n|^2 \leq \frac{C}{n^2 v^2}. \quad (4.17)$$

Proof. We write

$$\mathbf{E}|\Lambda_n|^2 = \mathbf{E}\Lambda_n \bar{\Lambda}_n = \mathbf{E} \frac{T_n}{z + m_n(z) + s(z)} \bar{\Lambda}_n = \sum_{\nu=1}^4 \mathbf{E} \frac{T_{n\nu}}{z + m_n(z) + s(z)} \bar{\Lambda}_n,$$

where

$$T_{n\nu} := \frac{1}{n} \sum_{j=1}^n \varepsilon_{j\nu} R_{jj}, \text{ for } \nu = 1, \dots, 4.$$

First we observe that by (3.3)

$$|T_{n4}| = \frac{1}{n} |m'_n(z)| \leq \frac{1}{nv} \operatorname{Im} m_n(z).$$

Hence $|z + m_n^{(j)}(z) + s(z)| \geq \operatorname{Im} m_n^{(j)}(z)$ and Jensen's inequality yields

$$|\mathbf{E} \frac{T_{n4}}{z + m_n(z) + s(z)} \bar{\Lambda}_n| \leq \frac{1}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2. \quad (4.18)$$

Furthermore, we represent T_{n1} as follows

$$T_{n1} = T_{n11} + T_{n12},$$

where

$$\begin{aligned} T_{n11} &= -\frac{1}{n} \sum_{j=1}^n \varepsilon_{j1} \frac{1}{z + m_n(z)}, \\ T_{n12} &= \frac{1}{n} \sum_{j=1}^n \varepsilon_{j1} \left(R_{jj} + \frac{1}{z + m_n(z)} \right). \end{aligned}$$

Using these notations we may write

$$V_1 := \mathbf{E} \frac{T_{n11}}{z + m_n(z) + s(z)} \bar{\Lambda}_n = -\mathbf{E} \frac{(\frac{1}{n} \sum_{j=1}^n \varepsilon_{j1})}{(z + m_n(z))(z + s(z) + m_n(z))} \bar{\Lambda}_n.$$

Applying the Cauchy – Schwartz inequality twice and using the definition of ε_{j1} , we get by [1, Lemma 7.5 inequality (7.9)]

$$|V_1| \leq \frac{1}{|z^2 - 4|^{\frac{1}{2}}} \mathbf{E}^{\frac{1}{4}} \left| \frac{1}{n\sqrt{n}} \sum_{j=1}^n X_{jj} \right|^4 \mathbf{E}^{\frac{1}{4}} \left| \frac{1}{z + m_n(z)} \right|^4 \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2. \quad (4.19)$$

By Rosenthal's inequality, we have, for $z \in \mathbb{G}$

$$|V_1| \leq \frac{C}{n|z^2 - 4|^{\frac{1}{2}}} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 \leq \frac{1}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2. \quad (4.20)$$

Using (2.26) we rewrite T_{n12} , obtaining

$$V_2 := \mathbf{E} \frac{T_{n12}}{z + m_n(z) + s(z)} \bar{\Lambda}_n = \frac{1}{n\sqrt{n}} \sum_{j=1}^n \mathbf{E} \frac{X_{jj} \varepsilon_j R_{jj}}{(z + m_n(z))(z + m_n(z) + s(z))} \bar{\Lambda}_n.$$

By the Cauchy – Schwartz inequality, using the definition of ε_j (see representation (2.1)), we obtain

$$|V_2| \leq \frac{1}{\sqrt{n}} \sum_{\nu=1}^4 \mathbf{E}^{\frac{1}{2}} \frac{\left| \frac{1}{n} \sum_{j=1}^n \varepsilon_{j\nu} X_{jj} R_{jj} \right|^2}{|z + m_n(z) + s(z)|^2 |z + m_n(z)|^2} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 =: \sum_{\nu=1}^4 V_{2\nu}. \quad (4.21)$$

For $\nu = 1$, we have

$$V_{21} \leq \frac{1}{n} \mathbf{E}^{\frac{1}{2}} \frac{\left| \frac{1}{n} \sum_{j=1}^n X_{jj}^2 R_{jj} \right|^2}{|z + m_n(z) + s(z)|^2 |z + m_n(z)|^2} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2.$$

Applying Hölder's inequality and [1, Lemma 7.5 inequality (7.9)], we arrive at

$$\begin{aligned} V_{21} &\leq \frac{1}{n\sqrt{|z^2 - 4|}} \mathbf{E}^{\frac{2}{4+\kappa}} \left(\frac{1}{n} \sum_{j=1}^n |X_{jj}|^{\frac{4+\kappa}{2}} \right)^2 \\ &\quad \times \mathbf{E}^{\frac{\kappa}{4(4+\kappa)}} \left(\frac{1}{n} \sum_{j=1}^n |R_{jj}|^{\frac{2(4+\kappa)}{\kappa}} \right)^4 \mathbf{E}^{\frac{\kappa}{4(4+\kappa)}} \frac{1}{|z + m_n(z)|^{\frac{8(4+\kappa)}{\kappa}}} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2. \end{aligned} \quad (4.22)$$

Observe that

$$\begin{aligned} \mathbf{E} \left(\frac{1}{n} \sum_{j=1}^n |X_{jj}|^{\frac{4+\kappa}{2}} \right)^2 &= \left(\frac{1}{n} \sum_{j=1}^n \mathbf{E} |X_{jj}|^{\frac{4+\kappa}{2}} \right)^2 + \mathbf{E} \left(\frac{1}{n} \sum_{j=1}^n (|X_{jj}|^{\frac{4+\kappa}{2}} - \mathbf{E} |X_{jj}|^{\frac{4+\kappa}{2}}) \right)^2 \\ &\leq 2\mu_{4+\kappa} + \frac{2}{n^2} \sum_{j=1}^n \mathbf{E} |X_{jj}|^{4+\kappa} \leq C. \end{aligned} \quad (4.23)$$

The last inequality, inequality (4.22), Theorem 2.1 and [1, Lemma 7.6] together imply

$$V_{21} \leq \frac{C}{n\sqrt{|z^2 - 4|}} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2. \quad (4.24)$$

Furthermore, for $\nu = 4$, by [1, Lemma 7.16], we have

$$V_{24} \leq \frac{1}{nv\sqrt{n}} \mathbf{E}^{\frac{1}{2}} \frac{\frac{1}{n} \sum_{j=1}^n |X_{jj}|^2 |R_{jj}|^2}{|z + m_n(z) + s(z)|^2 |z + m_n(z)|^2} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2.$$

Applying Hölder's inequality and [1, Lemma 7.5], we get

$$V_{24} \leq \frac{1}{nv\sqrt{n}\sqrt{|z^2 - 4|}} \mathbf{E}^{\frac{2}{4+\varkappa}} \left(\frac{1}{n} \sum_{j=1}^n |X_{jj}|^{\frac{4+\varkappa}{2}} \right) \mathbf{E}^{\frac{\varkappa}{4(4+\varkappa)}} \left(\frac{1}{n} \sum_{j=1}^n |R_{jj}|^{\frac{4(4+\varkappa)}{\varkappa}} \right) \\ \mathbf{E}^{\frac{\varkappa}{4(4+\varkappa)}} \frac{1}{|z + m_n(z)|^{\frac{4(4+\varkappa)}{\varkappa}}} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2.$$

Applying Theorem 2.1, we obtain

$$V_{24} \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2. \quad (4.25)$$

By Hölder's inequality, we have for $\nu = 2, 3$,

$$V_{2\nu} \leq \frac{1}{\sqrt{n}} \mathbf{E}^{\frac{2}{4+\varkappa}} \left(\frac{1}{n} \sum_{j=1}^n \frac{|\varepsilon_{j\nu}|^{\frac{4+\varkappa}{2}} |X_{jj}|^{\frac{4+\varkappa}{2}}}{|z + m_n(z) + s(z)|^{\frac{4+\varkappa}{2}}} \right) \mathbf{E}^{\frac{\varkappa}{4(4+\varkappa)}} \left(\frac{1}{n} \sum_{j=1}^n |R_{jj}|^{\frac{8(4+\varkappa)}{\varkappa}} \right) \\ \times \mathbf{E}^{\frac{\varkappa}{4(4+\varkappa)}} \frac{1}{|z + m_n(z)|^{\frac{8(4+\varkappa)}{\varkappa}}} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2.$$

Note that for $\nu = 2, 3$, r.v. X_{jj} doesn't depend on $\varepsilon_{j\nu}$ and on the σ -algebra $\mathfrak{M}^{(j)}$. We may write, using [1, inequality (7.42)],

$$\mathbf{E} \frac{|\varepsilon_{j\nu}|^{\frac{4+\varkappa}{2}} |X_{jj}|^{\frac{4+\varkappa}{2}}}{|z + m_n(z) + s(z)|^{\frac{4+\varkappa}{2}}} \leq C\sqrt{\mu_{4+\varkappa}} \mathbf{E} \frac{|\varepsilon_{j\nu}|^{\frac{4+\varkappa}{2}}}{|z + m_n^{(j)}(z) + s(z)|^{\frac{4+\varkappa}{2}}}.$$

Applying now Lemmas 4.4, 4.3 and [1, Lemma 7.5], we arrive at

$$V_{2\nu} \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2, \text{ for } \nu = 2, 3. \quad (4.26)$$

Inequalities (4.24), (4.25), (4.26) together imply

$$V_2 \leq \frac{C}{nv} \mathbf{E} |\Lambda_n|^2. \quad (4.27)$$

Consider now the quantity

$$Y_\nu := \mathbf{E} \frac{T_{n\nu}}{z + m_n(z) + s(z)} \bar{\Lambda}_n,$$

for $\nu = 2, 3$. We represent it as follows

$$Y_\nu = Y_{\nu 1} + Y_{\nu 2},$$

where

$$\begin{aligned} Y_{\nu 1} &= -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} \bar{\Lambda}_n}{(z + m_n^{(j)}(z))(z + m_n(z) + s(z))}, \\ Y_{\nu 2} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} (R_{jj} + \frac{1}{z + m_n^{(j)}(z)}) \bar{\Lambda}_n}{z + m_n(z) + s(z)}. \end{aligned}$$

By the representation (2.26), we have

$$Y_{\nu 2} = \sum_{\mu=1}^3 \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} \varepsilon_{j\mu} \bar{\Lambda}_n R_{jj}}{(z + m_n(z) + s(z))(z + m_n^{(j)}(z))}.$$

Using [1, inequality (7.42)], we may write, for $z \in \mathbb{G}$

$$|Y_{\nu 2}| \leq \sum_{\mu=1}^3 \frac{C}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu}| |\varepsilon_{j\mu}| |\bar{\Lambda}_n| |R_{jj}|}{|z + m_n^{(j)}(z) + s(z)| |z + m_n^{(j)}(z)|}.$$

Applying the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we get

$$\begin{aligned} |Y_{\nu 2}| &\leq \sum_{\mu=1}^3 \frac{C}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\mu}|^2 |R_{jj}|}{|z + m_n^{(j)}(z) + s(z)| |z + m_n^{(j)}(z)|} |\Lambda_n| \\ &\leq \sum_{\mu=1}^3 \frac{C}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\mu}|^2}{|z + m_n^{(j)}(z) + s(z)| |z + m_n^{(j)}(z)|} |\Lambda_n^{(j)}| |R_{jj}| \\ &\quad + \sum_{\mu=1}^3 \frac{C}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\mu}|^2}{|z + m_n^{(j)}(z) + s(z)| |z + m_n^{(j)}(z)|} |\varepsilon_{j4}| |R_{jj}| = \hat{Y}_1 + \hat{Y}_2. \end{aligned} \quad (4.28)$$

Applying Hölder's inequality, we get

$$\mathbf{E}\{|\varepsilon_{j\mu}|^2|R_{jj}|\mathfrak{M}^{(j)}\} \leq \mathbf{E}^{\frac{4}{4+\kappa}}\{|\varepsilon_{j\mu}|^{\frac{4+\kappa}{2}}|\mathfrak{M}^{(j)}\}\mathbf{E}^{\frac{\kappa}{4+\kappa}}\{|R_{jj}|^{\frac{4+\kappa}{\kappa}}|\mathfrak{M}^{(j)}\}. \quad (4.29)$$

By Lemmas 4.4 and 4.3, for $\mu = 1, 2, 3$, we have

$$\mathbf{E}\{|\varepsilon_{j\mu}|^2|R_{jj}|\mathfrak{M}^{(j)}\} \leq C\left(\frac{1}{n} + \frac{1}{n^2} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^2 + \frac{1}{nv} \operatorname{Im} m_n^{(j)}(z)\right) \mathbf{E}^{\frac{\kappa}{4+\kappa}}\{|R_{jj}|^{\frac{4+\kappa}{\kappa}}|\mathfrak{M}^{(j)}\}. \quad (4.30)$$

Conditioning and using inequality (4.30) and applying Theorem 2.1, we arrive at

$$\begin{aligned} \mathbf{E} \frac{|\varepsilon_{j\mu}|^2}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|} |\Lambda_n^{(j)}| |R_{jj}| &\leq \left(\frac{C}{n|z^2 - 4|^{\frac{1}{2}}} + \frac{C}{nv}\right) \mathbf{E}^{\frac{1}{2}} |\Lambda_n^{(j)}|^2 \\ &\leq \left(\frac{C}{n|z^2 - 4|^{\frac{1}{2}}} + \frac{C}{nv}\right) \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \left(\frac{C}{n^2 v |z^2 - 4|^{\frac{1}{2}}} + \frac{C}{(nv)^2}\right) \\ &\leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{(nv)^2}. \end{aligned} \quad (4.31)$$

The last inequality implies

$$|Y_{\nu 2}| \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{(nv)^2}. \quad (4.32)$$

In order to estimate $Y_{\nu 1}$ we introduce now the quantity

$$\Lambda_n^{(j1)} = \frac{1}{n} \operatorname{Tr} \mathbf{R}^{(j)} - s(z) + \frac{s(z)}{n} + \frac{1}{n^2} \operatorname{Tr} \mathbf{R}^{(j)2} s(z).$$

Recall that

$$\begin{aligned} \eta_{j1} &= \frac{1}{n} \sum_{l \in \mathbb{T}_j} [(\mathbf{R}^{(j)})^2]_{ll}, \quad \eta_{j2} = \frac{1}{n} \sum_{k \neq l \in \mathbb{T}_j} X_{jk} X_{jl} [(\mathbf{R}^{(j)})^2]_{l,k}, \\ \eta_{j3} &= \frac{1}{n} \sum_{l \in \mathbb{T}_j} (X_{jl}^2 - 1) [(\mathbf{R}^{(j)})^2]_{ll}. \end{aligned} \quad (4.33)$$

Note that

$$|\eta_{j1}| \leq \frac{1}{n} |\operatorname{Tr} (\mathbf{R}^{(j)})^2|. \quad (4.34)$$

We use that (see [1, equality (7.41)])

$$\varepsilon_{j4} = \frac{1}{n} (1 + \eta_{j1} + \eta_{j2} + \eta_{j3}) R_{jj}. \quad (4.35)$$

Note that

$$\begin{aligned}\delta_{nj} &= \Lambda_n - \tilde{\Lambda}_n^{(j)} = -\varepsilon_{j4} - \frac{s(z)}{n} - \frac{1}{n}\eta_{j0}s(z) \\ &= \frac{1}{n}(R_{jj} - s(z))(1 + \eta_{j1}) + \frac{1}{n}(\eta_{j2} + \eta_{j3})R_{jj}.\end{aligned}$$

This yields

$$|\delta_{nj}| \leq \frac{1}{n}(1 + |\eta_{j1}|)|R_{jj} - s(z)| + \frac{1}{n}|\eta_{j2} + \eta_{j3}||R_{jj}| \quad (4.36)$$

We represent $Y_{\nu 1}$ in the form

$$Y_{\nu 1} = Z_{\nu 1} + Z_{\nu 2} + Z_{\nu 3} + Z_{\nu 4},$$

where

$$\begin{aligned}Z_{\nu 1} &= -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} \bar{\Lambda}_n^{(j1)}}{(z + m_n^{(j)}(z))(z + m_n^{(j)}(z) + s(z))}, \\ Z_{\nu 2} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} \bar{\delta}_{nj}}{(z + m_n^{(j)}(z))(z + m_n(z) + s(z))}, \\ Z_{\nu 3} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} \bar{\Lambda}_n \varepsilon_{j4}}{(z + m_n^{(j)}(z))(z + m_n^{(j)}(z) + s(z))(z + m_n(z) + s(z))}, \\ Z_{\nu 4} &= -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} \bar{\delta}_{nj} \varepsilon_{j4}}{(z + m_n^{(j)}(z))(z + m_n^{(j)}(z) + s(z))(z + m_n(z) + s(z))}.\end{aligned}$$

First, note that by conditional independence

$$Z_{\nu 1} = 0. \quad (4.37)$$

Using the triangle inequality and [1, inequality (7.42)], we write

$$|Z_{\nu 3}| \leq \hat{Z}_{\nu 3} + \tilde{Z}_{\nu 3}, \quad (4.38)$$

where

$$\begin{aligned}\hat{Z}_{\nu 3} &= \frac{1}{n} \sum_{j=12}^n \mathbf{E} \frac{|\varepsilon_{j\nu}| |\Lambda_n^{(j)}| |\varepsilon_{j4}|}{|z + m_n^{(j)}| |z + m_n^{(j)}(z) + s(z)|^2}, \\ \tilde{Z}_{\nu 3} &= \frac{1}{n} \sum_{j=12}^n \mathbf{E} \frac{|\varepsilon_{j\nu}| |\varepsilon_{j4}|^2}{|z + m_n^{(j)}| |z + m_n^{(j)}(z) + s(z)|^2}\end{aligned} \quad (4.39)$$

Using that $|\varepsilon_{j4}| \leq 1/nv$ and applying the Cauchy – Schwartz inequality, we get

$$\tilde{Z}_{\nu 3} \leq \frac{1}{nv} \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} \left(\frac{|\varepsilon_{j\nu}|^2}{|z + m_n^{(j)}|^2 |z + m_n^{(j)}(z) + s(z)|^2} \right) \mathbf{E}^{\frac{1}{2}} \left(\frac{|\varepsilon_{j4}|^2}{|z + m_n^{(j)}(z) + s(z)|^2} \right). \quad (4.40)$$

Conditioning and applying Theorem 2.1 and [1, Lemma 7.5 inequality (7.0)], we obtain, for $z \in \mathbb{G}$,

$$\begin{aligned} & \mathbf{E}^{\frac{1}{2}} \left(\frac{|\varepsilon_{j\nu}|^2}{|z + m_n^{(j)}|^2 |z + m_n^{(j)}(z) + s(z)|^2} \right) \\ & \leq \frac{C}{\sqrt{nv}} \mathbf{E}^{\frac{1}{2}} \left(\frac{1}{|z + m_n^{(j)}|^2 |z + m_n^{(j)}(z) + s(z)|} \right) \leq \frac{C}{\sqrt{nv} |z^2 - 4|^{\frac{1}{4}}} \leq C. \end{aligned} \quad (4.41)$$

According to Lemma 4.2, we have

$$\mathbf{E}^{\frac{1}{2}} \left(\frac{|\varepsilon_{j4}|^2}{|z + m_n^{(j)}(z) + s(z)|^2} \right) \leq \frac{C}{nv}. \quad (4.42)$$

The last inequalities together imply, for $z \in \mathbb{G}$

$$\tilde{Z}_{\nu 3} \leq \frac{C}{n^2 v^2} \quad (4.43)$$

Furthermore, conditioning and applying the Cauchy – Schwartz inequality, we obtain

$$\begin{aligned} \mathbf{E}\{|\varepsilon_{j4}| |\varepsilon_{j\nu}| \left| \mathfrak{M}^{(j)} \right\} & \leq \mathbf{E}^{\frac{1}{2}} \{|\varepsilon_{j\nu}|^2 \left| \mathfrak{M}^{(j)} \right\} \mathbf{E}^{\frac{1}{2}} \{|\varepsilon_{j4}|^2 \left| \mathfrak{M}^{(j)} \right\} \\ & \leq \frac{1}{\sqrt{nv}} (v + \operatorname{Im} m_n^{(j)}(z))^{\frac{1}{2}} \mathbf{E}^{\frac{1}{2}} \{|\varepsilon_{j4}|^2 \left| \mathfrak{M}^{(j)} \right\}. \end{aligned} \quad (4.44)$$

Applying the Cauchy – Schwartz inequality again and using [1, Lemma 7.5 inequality (7.9)], we get

$$\hat{Z}_{\nu 3} \leq \frac{C}{\sqrt{nv} |z^2 - 4|^{\frac{1}{4}}} \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{4}} \frac{|\varepsilon_{j4}|^2}{|z + m_n^{(j)}(z)|^2 |z + m_n^{(j)}(z) + s(z)|^2} \mathbf{E}^{\frac{1}{2}} |\Lambda_n^{(j)}|^2.$$

The last inequality and Lemma 4.2 with $\theta = 2$ imply that, for $z \in \mathbb{G}$,

$$\hat{Z}_{\nu 3} \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{n^2 v^2}. \quad (4.45)$$

Furthermore, note that

$$|1 + \eta_{j1}| \leq v^{-1} \operatorname{Im} \{z + m_n^{(j)}(z)\} \leq \operatorname{Im} \{z + m_n^{(j)}(z) + s(z)\}.$$

This inequality together with (4.36) implies that

$$|Z_{\nu 4}| \leq \tilde{Z}_{\nu 4} + \hat{Z}_{\nu 4}, \quad (4.46)$$

where

$$\begin{aligned} \tilde{Z}_{\nu 4} &= \frac{1}{n^2 v} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu} \varepsilon_{j4}| |R_{jj} - s(z)|}{|z + m_n^{(j)}(z)| |z + m_n(z) + s(z)|}, \\ \hat{Z}_{\nu 4} &= \frac{1}{n^2 v} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu} \varepsilon_{j4}| |\eta_{j2} + \eta_{j3}| |R_{jj} - s(z)|}{|z + m_n^{(j)}(z)| |z + m_n(z) + s(z)|} \end{aligned} \quad (4.47)$$

By representation (3.2), we have

$$|R_{jj} - s(z)| \leq |\Lambda_n| |R_{jj}| + |\varepsilon_j| |R_{jj}|.$$

This implies that

$$\tilde{Z}_{\nu 4} \leq \tilde{Z}_{\nu 41} + \tilde{Z}_{\nu 42}, \quad (4.48)$$

where

$$\begin{aligned} \tilde{Z}_{\nu 41} &= \frac{1}{n^2 v} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu} \varepsilon_{j4}| |\Lambda_n| |R_{jj}|}{|z + m_n^{(j)}(z)| |z + m_n(z) + s(z)|}, \\ \tilde{Z}_{\nu 42} &= \frac{1}{n^2 v} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu} \varepsilon_{j4}| |\varepsilon_j| |R_{jj}|}{|z + m_n^{(j)}(z)| |z + m_n(z) + s(z)|}. \end{aligned}$$

Similar to the bound of $\hat{Z}_{\nu 3}$ we obtain

$$\tilde{Z}_{\nu 41} \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{(nv)^2}. \quad (4.49)$$

Note that

$$|\varepsilon_{j\nu} \varepsilon_{j4}| |\varepsilon_j| \leq 2|\varepsilon_{j1}|^2 |\varepsilon_{j4}| + 2|\varepsilon_{j2}|^2 |\varepsilon_{j4}| + 2|\varepsilon_{j3}|^2 |\varepsilon_{j4}| + 2|\varepsilon_{j4}|^3. \quad (4.50)$$

We may write now

$$\tilde{Z}_{\nu 42} \leq \check{Z}_1 + \cdots + \check{Z}_4, \quad (4.51)$$

where, for $\mu = 1, 2, 3, 4$

$$\check{Z}_\mu = \frac{1}{n^2 v} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\mu}|^2 |\varepsilon_{j4}| |R_{jj}|}{|z + m_n^{(j)}(z)| |z + m_n(z) + s(z)|}. \quad (4.52)$$

Using that $|\varepsilon_{j4}| \leq 1/(nv)$ and [1, inequality (7.42)], we obtain

$$\begin{aligned} \check{Z}_\mu &\leq \frac{C}{n^2 v^2} \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\mu}|^2 |R_{jj}|}{|z + m_n^{(j)}(z)| |z + m_n^{(j)}(z) + s(z)|} \\ &\leq \frac{C}{n^2 v^2} \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{4}{4+\varkappa}} \frac{|\varepsilon_{j\mu}|^{\frac{4+\varkappa}{4}}}{|z + m_n^{(j)}(z) + s(z)|^{\frac{4+\varkappa}{4}}} \mathbf{E}^{\frac{\varkappa}{2(4+\varkappa)}} |R_{jj}|^{\frac{2(4+\varkappa)}{\varkappa}} \\ &\quad \times \mathbf{E}^{\frac{\varkappa}{2(4+\varkappa)}} \frac{1}{|z + m_n^{(j)}(z)|^{\frac{2(4+\varkappa)}{\varkappa}}}. \end{aligned} \quad (4.53)$$

Applying Lemmas 4.4, 4.3, 4.2, we obtain for $z \in \mathbb{G}$,

$$\check{Z}_\mu \leq \frac{C}{(nv)^2}. \quad (4.54)$$

This implies that

$$\tilde{Z}_{\nu 42} \leq \frac{C}{(nv)^2}. \quad (4.55)$$

Inequalities (4.49) and (4.55) together imply

$$|Z_{\nu 4}| \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{n^2 v^2}. \quad (4.56)$$

To bound $Z_{\nu 2}$ we first apply [1, inequality (7.42)] and obtain

$$|Z_{\nu 2}| \leq \frac{C}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu}| |\delta_{nj}|}{|z + m_n^{(j)}(z)| |z + m_n^{(j)}(z) + s(z)|}.$$

Furthermore, similarly to the bound of $Z_{\nu 4}$ – inequality (4.48) – we may write

$$|Z_{\nu 2}| \leq \tilde{Z}_{\nu 2} + \hat{Z}_{\nu 2},$$

where

$$\begin{aligned} \tilde{Z}_{\nu 2} &= \frac{C}{n^2} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu}| |R_{jj} - s(z)|}{|z + m_n^{(j)}(z)|}, \\ \hat{Z}_{\nu 2} &= \frac{C}{n^2} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu}| |\eta_{j2} + \eta_{j3}| |R_{jj}|}{|z + m_n^{(j)}(z)| |z + m_n^{(j)}(z) + s(z)|}. \end{aligned}$$

Furthermore,

$$\tilde{Z}_{\nu 2} \leq \frac{C}{n^2} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu}| |\varepsilon_j| |R_{jj}|}{|z + m_n^{(j)}(z)|} + \frac{C}{n^2} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu}| |\Lambda_n| |R_{jj}|}{|z + m_n^{(j)}(z)|}. \quad (4.57)$$

Lemmas 7.15, 7.16, 7.22, inequality (7.39) and Corollary 5.14 together imply

$$\tilde{Z}_{\nu 2} \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{n^2 v^2}. \quad (4.58)$$

Conditioning and applying now Hölder's inequality, we get

$$|\hat{Z}_{\nu 2}| \leq \frac{C}{n^2 v^2} \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{\kappa}{4+\kappa}} |R_{jj}|^{\frac{4+\kappa}{\kappa}} \mathbf{E}^{\frac{4}{4+\kappa}} \frac{1}{|z + m_n^{(j)}(z)|^{\frac{4+\kappa}{4}}}.$$

The last inequality together with Theorem 2.1 implies

$$|\hat{Z}_{\nu 2}| \leq \frac{C}{n^2 v^2}. \quad (4.59)$$

Combining inequalities (4.18), (4.20), (4.27), (4.32), (4.37), (4.45), (4.54), (4.55), (4.56), (4.58) and (4.59), we get

$$\mathbf{E} |\Lambda_n|^2 \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{n^2 v^2}. \quad (4.60)$$

Applying [1, Lemma 7.4] with $t = 2$ and $r = 1$ completes the proof of Lemma 4.5. □

References

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